Problem 12.3. If we record who selects which card there are 52 options for the card of the first person, then there are 51 options for the card of the second person and lastly 50 options for the card of the third person. So there are $\boxed{52 \cdot 51 \cdot 50}$ options in total.

If we don't record who selects which card the question is how many triples of 3 cards there are. By definition, this number is $\boxed{\binom{52}{3}}$.

Problem 12.5. Since $(1+x)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i} x^i$, the coefficient of x^n on the left equals $\binom{2n}{n}$. On the other hand,

$$(1+x)^{2n} = (1+x)^n \cdot (1+x)^n = \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right) \cdot \left(\sum_{j=0}^{n} \binom{n}{j} x^j\right).$$

Now, when expanding this product, and we consider a summand $\binom{n}{i}x^i$ from the first factor, then the only summand from the second factor that leads to x^n is $\binom{n}{n-i}x^{n-i}$, so this leads to a coefficient $\binom{n}{i}\binom{n}{n-i}$ for x^n. Since i can range from 0 to n the coefficient of x^n on the left equals $\sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i}$.

Now, using that $\binom{n}{n-i} = \binom{n}{i}$ we find

$$\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i}\binom{n}{n-i} = \sum_{i=0}^{n} \binom{n}{i}^2.$$

Problem III.16. Let A_i be the sets defined in the problem. Note that if $f : \mathbb{N}_m \to \mathbb{N}_n$ is a surjection, then for each $i \in \mathbb{N}_n$ we know that f attains the value i, which means that $f \notin A_i$, so if f is a surjection then $f \notin A_1, A_2, \ldots, A_n$. On the other hand, if $f \notin A_i$, then f must attain the value i, so if $f \notin A_1, A_2, \ldots, A_n$, then f must attain the values $1, 2, \ldots, n$, so f is a surjection. Therefore,

$$f \text{ is a surjection} \iff f \notin A_1, A_2, \ldots, A_n \iff f \notin A_1 \cup A_2 \cup \ldots \cup A_n.$$

Therefore, the number of surjections equals the total number of functions $\mathbb{N}_m \to \mathbb{N}_n$ minus the number of functions in $A_1 \cup A_2 \cup \ldots \cup A_n$. Since the total number of functions equals n^m we find that the number of surjections from $\mathbb{N}_m$ to $\mathbb{N}_n$ is given by

$$n^m - |A_1 \cup A_2 \cup \ldots \cup A_n| = n^m - \sum_{\emptyset \neq I \subseteq \mathbb{N}_n} (-1)^{|I|-1} |A_I|.$$

Suppose that $I = \{i_1, \ldots, i_k\}$. Then $A_I = A_{i_1} \cap \ldots \cap A_{i_k} = \{f : \mathbb{N}_m \to \mathbb{N}_n | f \text{ does not attain the values } i_1, \ldots, i_k\}$. Therefore, the functions in A_I are exactly the functions $f : \mathbb{N}_m \to \mathbb{N}_n \backslash \{i_1, \ldots, i_k\}$, i.e. the functions $\mathbb{N}_m \to \mathbb{N}_n \backslash I$. Since $|\mathbb{N}_n \backslash I| = |\mathbb{N}_n| - |I| = n - |I|$ the number of such functions equals $(n - |I|)^m$. Therefore, the desired number of surjections equals

$$\begin{aligned} n^m - \sum_{\emptyset \neq I \subseteq \mathbb{N}_n} (-1)^{|I|-1} |A_I| &= n^m - \sum_{k=1}^{n} \sum_{\substack{I \subseteq \mathbb{N}_n \\ |I|=k}} (-1)^{|I|-1} |A_I| = n^m - \sum_{k=1}^{n} \sum_{\substack{I \subseteq \mathbb{N}_n \\ |I|=k}} (-1)^{k-1} (n-k)^m \\ &= n^m - \sum_{k=1}^{n} \binom{n}{k} (-1)^{k-1} (n-k)^m = n^m + \sum_{k=1}^{n} (-1)^k \binom{n}{k} (n-k)^m, \end{aligned}$$

since there are $\binom{n}{k}$ subsets $I \subseteq \mathbb{N}_n$ of size exactly k. Now, this is exactly the expression we needed to show.

Using this formula for $m = n$ we find that the number of surjections $\mathbb{N}_n \to \mathbb{N}_n$ equals

$$n^n - \binom{n}{1}(n-1)^n + \binom{n}{2}(n-2)^n - \ldots + (-1)^{n-1}\binom{n}{n-1}1^n.$$

On the other hand, a function $\mathbb{N}_n \to \mathbb{N}_n$ is a surjection if and only if it is a bijection, and we know that there are $n!$ such functions, so the above expression must be equal to $n!$.

Problem III.18. Define the function

$$f : \bigcup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y) \to \mathcal{P}_k(X \cup Y)$$

by $f(A, B) = A \cup B$. Let us first check that this function is well defined. First, since $(A, B) \in \cup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)$ there exists some $0 \leq j \leq k$ with $(A, B) \in \mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y)$, in other words $A \subseteq X$, $B \subseteq Y$ and $|A| = j$ and $|B| = k - j$. Now, since $A \subseteq X$ and $B \subseteq Y$ we clearly have $A \cup B \subseteq X \cup Y$. Furthermore, since X and Y are disjoint, A and B are also disjoint, so $|A \cup B| = |A| + |B| - |A \cap B| = j + (k - j) - 0 = k$, so $A \cup B \in \mathcal{P}_k(X \cup Y)$.

Now, we show that f is a bijection. We will show this by finding its inverse. Let g be the function

$$g : \mathcal{P}_k(X \cup Y) \to \bigcup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)$$

defined by $g(C) = (C \cap X, C \cap Y)$. We will show that g is the inverse of f. Let us first check that g is well defined. Firstly notice that $C \cap X \subseteq X$ and $C \cap Y \subseteq Y$. Now suppose that $|C \cap X| = j$. Since X and Y are disjoint, $C \cap X$ and $C \cap Y$ are also disjoint. Furthermore, $(C \cap X) \cup (C \cap Y) = C \cap (X \cup Y) = C$ since $C \subseteq X \cup Y$. Therefore, $|C \cap Y| = |C| - |C \cap X| = k - j$. Therefore, $(C \cap X, C \cap Y) \in \mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y) \subseteq \cup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)$.

Now, suppose that $(A, B) \in \cup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)$, then

$$\begin{aligned}(g \circ f)(A, B) = g(f(A, B)) = g(A \cup B) &= ((A \cup B) \cap X, (A \cup B) \cap Y) \\ &= ((A \cap X) \cup (B \cap X), (A \cap Y) \cup (B \cap Y)) = (A \cup \emptyset, \emptyset \cup B) = (A, B),\end{aligned}$$

where we use that $A \subseteq X$, so $A \cap X = A$ and $A \cap Y = \emptyset$ (as X and Y are disjoint) and similarly for B.

Also, if $C \in \mathcal{P}_k(X \cup Y)$ then

$$(f \circ g)(C) = f(g(C)) = f(C \cap X, C \cap Y) = (C \cap X) \cup (C \cap Y) = C \cap (X \cup Y) = C,$$

since $C \subseteq X \cup Y$. This shows that g is indeed the inverse of f, so f is a bijection.

Now, take $X = \{1, 2, \ldots, n\}$ and $Y = \{n + 1, \ldots, n + m\}$, such that $X \cup Y = \{1, 2, \ldots, n + m\}$. Note that $\cup_{i=0}^{k} \mathcal{P}_i(X) \times \mathcal{P}_{k-i}(Y)$ is in fact a disjoint union. Indeed, if a pair (A, B) belongs to the union, then it belongs to $\mathcal{P}_j(X) \times \mathcal{P}_{k-j}(Y)$ for $j = |A|$ and no other index. Now, since sets that admit a bijection between them have equal cardinality we conclude that

$$\begin{aligned}\binom{n+m}{k} = |\mathcal{P}_k(\{1,2,\ldots,n+m\})| &= \left|\bigcup_{i=0}^{k} \mathcal{P}_i(\{1,2,\ldots,n\}) \times p_{k-i}(\{n+1,\ldots,n+m\})\right| \\ &= \sum_{i=0}^{k} |\mathcal{P}_i(\{1,2\ldots,n\}) \times \mathcal{P}_{k-i}(\{n+1,\ldots,n+m\})| \\ &= \sum_{i=0}^{k} |\mathcal{P}_i(\{1,2,\ldots,n\})| \cdot |\mathcal{P}_{k-i}(\{n+1,\ldots,n+m\})| = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}.\end{aligned}$$

Now we will prove this last equality by induction. Let $P(n)$ be the statement

$$P(n): \binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} \text{ for all } m,k.$$

Base case. For $n=0$ we have, for any m,k, that $\sum_{i=0}^{k}\binom{m}{i}\binom{0}{k-i} = 0+0+\ldots+0+\binom{m}{k}\cdot 1 = \binom{m}{k}$, since $\binom{0}{t} = 0$ when $t>0$.
Induction hypothesis. Assume that $P(\ell)$ is true for some integer $\ell \geq 0$.
Induction step. Let m,k be any positive integers. Then

$$\begin{aligned}\binom{m+\ell+1}{k} &= \binom{m+\ell}{k} + \binom{m+\ell}{k-1} \\ &= \sum_{i=0}^{k}\binom{m}{i}\binom{\ell}{k-i} + \sum_{i=0}^{k-1}\binom{m}{i}\binom{\ell}{(k-1)-i},\end{aligned}$$

where we used the induction hypothesis twice. Note that this is the reason we included the "for all m,k" part in the statement $P(n)$, since this is what allows us to use the induction hypothesis for the second binomial coefficient (where we use it for $k-1$ instead of k). Now, this rewrites as

$$\begin{aligned}\binom{m+\ell+1}{k} &= \left(\sum_{i=0}^{k-1}\binom{m}{i}\binom{\ell}{k-i}\right) + \binom{m}{k} + \sum_{i=0}^{k-1}\binom{m}{i}\binom{\ell}{k-i-1} \\ &= \sum_{i=0}^{k-1}\binom{m}{i}\left(\binom{\ell}{k-i} + \binom{\ell}{k-i-1}\right) + \binom{m}{k}\binom{\ell+1}{0} \\ &= \sum_{i=0}^{k-1}\binom{m}{i}\binom{\ell+1}{k-i} + \binom{m}{k}\binom{\ell+1}{0} = \sum_{i=0}^{k}\binom{m}{i}\binom{\ell+1}{k-i},\end{aligned}$$

as desired.

Problem 1. By **Theorem 10.2.3** we know that $|Y \times Z| = bc$. Now, by **Proposition 12.1.4** the number of injections $X \to Y \times Z$ is equal to $bc(bc-1)\cdots(bc-a+1)$.

Problem 2. There is a purely algebraic proof of this fact by noticing that

$$\frac{n}{k}\binom{n-1}{k-1} = \frac{n}{k}\cdot\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} = \frac{n}{k}\cdot\frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n!}{k!(n-k)!} = \binom{n}{k}.$$

However, there is also a nice combinatorial proof using the definition of binomial coefficients. The equation we want to show is $k\binom{n}{k} = n\binom{n-1}{k-1}$. Now, remember that $\binom{a}{b}$ is the number of ways to select b elements from a set of a elements. Now, suppose that we have a set of n books, from which we want to pick k books to take with us during summer break and we want to select one book as the first book we will read. We can count the number of ways we can do this in two ways.

Firstly, we can first select the k books in $\binom{n}{k}$ ways. After that, there are k ways to pick the first book to read, so there are $k \cdot \binom{n}{k}$ ways in total.

On the other hand, we can also first pick the book we will read first, which we can do in n ways. After that, we have to choose the $k-1$ other books from the remaining $n-1$ books, which we can do in $\binom{n-1}{k-1}$ ways. So in total there are $n \cdot \binom{n-1}{k-1}$ ways.

Since both numbers still count the same thing, we must have $k \cdot \binom{n}{k} = n \cdot \binom{n-1}{k-1}$.

Problem 3. This is just a special case of **Problem III.18.** by taking $k = n$, $X = \{1, 2, \ldots, n\}$ and $Y = \{n+1, \ldots, 2n\}$.